MATH 2028 - Integrability Criteria

GOAL: Find necessary & sufficient criteria for a bdd function $f : R \subseteq \mathbb{R}^n \to \mathbb{R}$ to be integrable.

Prop: (Riemann condition)

Let $f : R \to \mathbb{R}$ be a bdd function defined on a rectangle $R \subseteq \mathbb{R}^n$. Then, f is integrable over R IF AND ONLY IF $\forall \varepsilon > 0$, $\exists$ partition $\mathcal{P}$ of R s.t. $\boxed{U(f, \mathcal{P}) - L(f, \mathcal{P}) < \varepsilon.}$

Proof: "$\Rightarrow$" Suppose f is integrable over R.

THEN: $$\sup_{\mathcal{P}} L(f, \mathcal{P}) = \int_R f \, dV = \inf_{\mathcal{P}} U(f, \mathcal{P}).$$

By def$^{\underline{n}}$ of sup & inf, $\forall \varepsilon > 0$, $\exists$ partitions $\mathcal{P}', \mathcal{P}''$

s.t. $$\int_R f \, dV - \frac{\varepsilon}{2} < L(f, \mathcal{P}')$$

and $$\int_R f \, dV + \frac{\varepsilon}{2} > U(f, \mathcal{P}'')$$

Let $\mathcal{P}$ be a common refinement of $\mathcal{P}'$ and $\mathcal{P}''$.

By a previous lemma.

$$\int_R f\,dV - \frac{\varepsilon}{2} < L(f,\mathcal{P}') \leq L(f,\mathcal{P})$$

$$\int_R f\,dV + \frac{\varepsilon}{2} > U(f,\mathcal{P}'') \geq U(f,\mathcal{P})$$

Therefore, $U(f,\mathcal{P}) - L(f,\mathcal{P}) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$.

"$\Leftarrow$" Suppose $\forall \varepsilon > 0$, $\exists$ partition $\mathcal{P}$ of R

s.t. $U(f,\mathcal{P}) - L(f,\mathcal{P}) < \varepsilon$ ······· $(*)$

Assume on the contrary that f is <u>NOT</u> integrable over R. Then

$$I_1 := \sup_{\mathcal{P}} L(f,\mathcal{P}) < \inf_{\mathcal{P}} U(f,\mathcal{P}) =: I_2$$

Choose $\varepsilon := \frac{1}{2}(I_2 - I_1) > 0$. then for <u>ANY</u> partition $\mathcal{P}$ of R, we have

$$U(f,\mathcal{P}) - L(f,\mathcal{P})$$

$$\geq \inf_{\mathcal{P}} U(f,\mathcal{P}) - \sup L(f,\mathcal{P}) = I_2 - I_1 > \varepsilon$$

which is a contradiction to $(*)$.

$\blacksquare$

The following two propositions provide a way to generate new integrable functions.

<u>Prop</u>: Let $f, g : R \to \mathbb{R}$ be bdd integrable functions over a rectangle $R \subseteq \mathbb{R}^n$. THEN, $f \pm g$ and αf are integrable over R. $\forall \alpha \in \mathbb{R}$.

Moreover, we have

$$\boxed{\int_R (f \pm g)\, dV = \int_R f\, dV \pm \int_R g\, dV}$$

and
$$\boxed{\int_R \alpha f\, dV = \alpha \int_R f\, dV}$$

<u>Prop</u>: Let $f : R \to \mathbb{R}$ be a bdd function on a rectangle $R \subseteq \mathbb{R}^n$. Suppose $R = \underbrace{R_1 \cup R_2}_{\mathrm{int}(R_1) \cap \mathrm{int}(R_2) = \phi}$ for some rectangles $R_1, R_2 \subseteq \mathbb{R}^n$. THEN,

f is integrable on R $\iff$ $f|_{R_i} : R_i \to \mathbb{R}$, $i = 1, 2$, is integrable on R_i

Moreover, $$\boxed{\int_{R_1 \cup R_2} f\, dV = \int_{R_1} f\, dV + \int_{R_2} f\, dV}$$

<u>Proof</u>: Exercises.

So far we have not seen many examples of integrable functions. The following proposition, however, shows that integrable functions exist in abundance.

<u>Prop</u>: Any continuous $f: R \to \mathbb{R}$ on a rectangle $R \subseteq \mathbb{R}^n$ is integrable.

<u>Proof</u>: We want to apply Riemann condition, i.e.

<u>Claim</u>: $\forall \varepsilon > 0$, $\exists \mathcal{P}$ s.t. $\mathcal{U}(f, \mathcal{P}) - L(f, \mathcal{P}) < \varepsilon$

We shall make use of a fact from analysis:

<u>FACT</u>: Any continuous function on a compact set is "uniformly continuous"

Hence, $\forall \varepsilon' > 0$, $\exists \delta > 0$ s.t. (**)

$$\begin{array}{c} \|x-y\| < \delta \\ x, y \in R \end{array} \implies |f(x) - f(y)| < \varepsilon'$$

<u>Proof of Claim</u>: Fix any $\varepsilon > 0$, choose $\varepsilon' = \dfrac{\varepsilon}{\mathrm{Vol}(R)} > 0$

and then $\delta > 0$ as above. Then, we fix a partition $\mathcal{P}$ of R s.t. $\forall Q \in \mathcal{P}$,

$$\operatorname{diam}(Q) := \sup_{x,y\in Q} \|x-y\| < \delta$$

By $(**)$, for any $Q \in \mathcal{P}$,

$$\sup_{x\in Q} f(x) - \inf_{x\in Q} f(x) < \frac{\varepsilon}{\operatorname{Vol}(R)}$$

Thus, we have

$$U(f,\mathcal{P}) - L(f,\mathcal{P})$$

$$= \sum_{Q\in\mathcal{P}} \left(\sup_{x\in Q} f(x) - \inf_{x\in Q} f(x) \right) \cdot \operatorname{Vol}(Q)$$

$$< \frac{\varepsilon}{\operatorname{Vol}(R)} \sum_{Q\in\mathcal{P}} \operatorname{Vol}(Q) = \varepsilon .$$

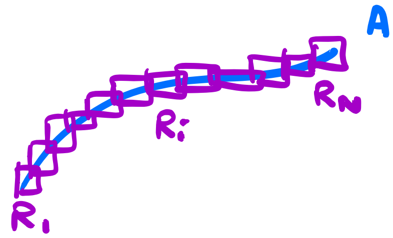
$\square$

It turns out that a bdd discontinuous function can still be integrable as long as the set of discontinuities is "small" in some sense.

Def$^{\underline{n}}$: A subset $A \subseteq \mathbb{R}^n$ is said to have **content zero** if $\forall\, \varepsilon > 0$, $\exists$ finitely many rectangles $R_1, \dots, R_N$ s.t.

(i) $A \subseteq R_1 \cup \cdots \cup R_N$

(ii) $\sum_{i=1}^{N} \mathrm{Vol}(R_N) < \varepsilon$



Prop: Let $f : R \to \mathbb{R}$ be bdd on a rectangle $R \subseteq \mathbb{R}^n$, and $A := \{ x \in R \mid f \text{ is } \underline{\underline{\text{NOT}}} \text{ cts at } x \}$

If A has content zero, then f is integrable.

Proof: Again, we shall apply Riemann's condition.

Let $\varepsilon > 0$ be fixed. We want to find a partition $\mathcal{P}$ s.t. $U(f, \mathcal{P}) - L(f, \mathcal{P}) < \varepsilon$.

- f bdd $\Rightarrow \exists\, M > 0$ s.t. $|f(x)| \leq M$, $\forall\, x \in R$
- A has content zero $\Rightarrow \exists$ rectangles $R_1, \dots, R_N$ s.t.

(i) $A \subseteq R_1 \cup \cdots \cup R_N \subseteq R$

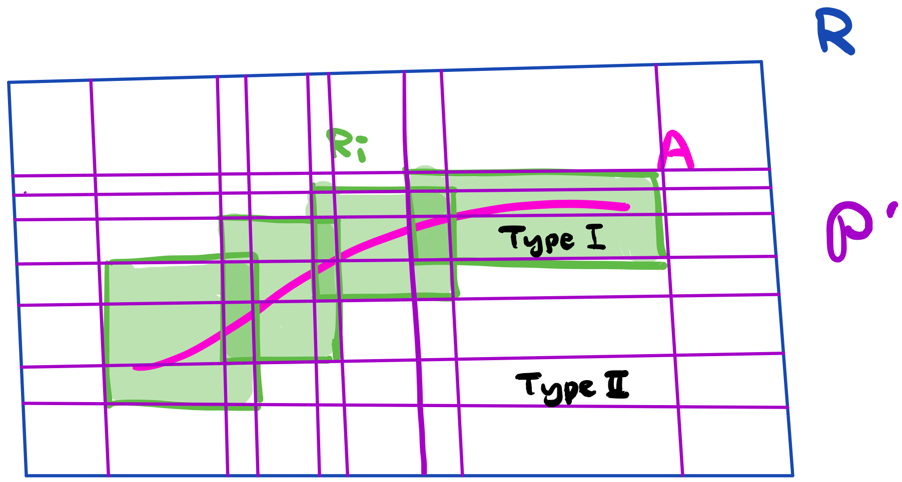
by taking $R_i \cap R$ otherwise

(ii) $\sum_{i=1}^{N} \mathrm{Vol}(R_i) < \dfrac{\varepsilon}{4M}$

By enlarging the rectangles R_i slightly if needed, we can assume WLOG that

the "boundary" of R_i w.r.t. R → $\boxed{\partial R_i} \cap A = \phi$ for $i = 1, \dots, N$

- Choose a partition $\mathcal{P}'$ of R s.t. each R_i is the union of some rectangles in $\mathcal{P}'$



- Any rectangle $Q' \in \mathcal{P}'$ belongs to exactly one of the two types:

<u>Type I</u> : $Q' \subseteq R_i$ for some i

<u>Type II</u> : $Q' \subseteq$ the closure of $R \setminus \bigcup_{i=1}^{N} R_i$

Note that for each Q' in Type II,

$f|_{Q'}$ is a cts function on Q'

hence f is integrable on Q' by previous proposition. Therefore, $\exists$ partition $\mathcal{P}_{Q'}$ of Q'

s.t. $U(f, \mathcal{P}_{Q'}) - L(f, \mathcal{P}_{Q'}) < \dfrac{\varepsilon}{2 \cdot \#\{\text{Type II } Q' \in \mathcal{P}'\}}$

- Take $\mathcal{P}$ as a partition of R which is a common refinement to ALL $\mathcal{P}_{Q'}$ above.

Then, we have

$$U(f, \mathcal{P}) - L(f, \mathcal{P})$$

$$= \sum_{\substack{Q \in \mathcal{P} \\ Q \subseteq Q' \text{ Type I}}} \left(\sup_{x \in Q} f(x) - \inf_{x \in Q} f(x) \right) \cdot \mathrm{vol}(Q)$$

$$+ \sum_{\substack{Q \in \mathcal{P} \\ Q \subseteq Q' \text{ Type II}}} \left(\sup_{x \in Q} f(x) - \inf_{x \in Q} f(x) \right) \cdot \mathrm{vol}(Q)$$

$$\leq 2M \cdot \sum_{i=1}^{N} \mathrm{vol}(R_i) + \sum_{\substack{Q' \in \mathcal{P}' \\ \text{Type II}}} U(f, \mathcal{P}_{Q'}) - L(f, \mathcal{P}_{Q'})$$

$$< 2M \cdot \frac{\varepsilon}{4M} + \frac{\varepsilon}{2} = \varepsilon$$

$\blacksquare$

Therefore, we have a "sufficient condition" for integrability:

$\boxed{f \text{ cts on } R \text{ } \underline{\text{except}} \text{ on a set of content zero}} \Longrightarrow \boxed{f \text{ integrable on } R}$

Nonetheless, this condition is <u>NOT</u> "necessary" as shown by the following

<u>Example</u>: (Thomae's function)

The function $f : [0,1] \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} 0 & \text{if } x \in \mathbb{Q}^c \\ \frac{1}{q} & \text{if } x = \frac{p}{q} \in \mathbb{Q} \text{ where } \\ & p, q \in \mathbb{N} \text{ are coprime} \end{cases}$$

is integrable but discontinuous on $\mathbb{Q} \cap [0,1]$.

To obtain a necessary <u>AND</u> sufficient condition for integrability, we need the notion of a "measure zero" subset.

Def$^{\underline{n}}$: A subset $A \subseteq \mathbb{R}^n$ is said to have measure zero if $\forall \varepsilon > 0$, $\exists$ a sequence of rectangles $\{R_i\}_{i=1}^{\infty}$ s.t.

(i) $A \subseteq \bigcup_{i=1}^{\infty} R_i$

(ii) $\sum_{i=1}^{\infty} \mathrm{Vol}(R_N) < \varepsilon$

The following theorem says precisely when a bdd function $f: R \to \mathbb{R}$ is integrable. The proof is rather involved and is left as a (challenging) exercise for the interested students.

Thm: Let $f: R \to \mathbb{R}$ be bdd on a rectangle $R \subseteq \mathbb{R}^n$, Then, we have

$\boxed{f \text{ cts on } R \text{ except on a set of measure zero}} \iff \boxed{f \text{ integrable on } R}$